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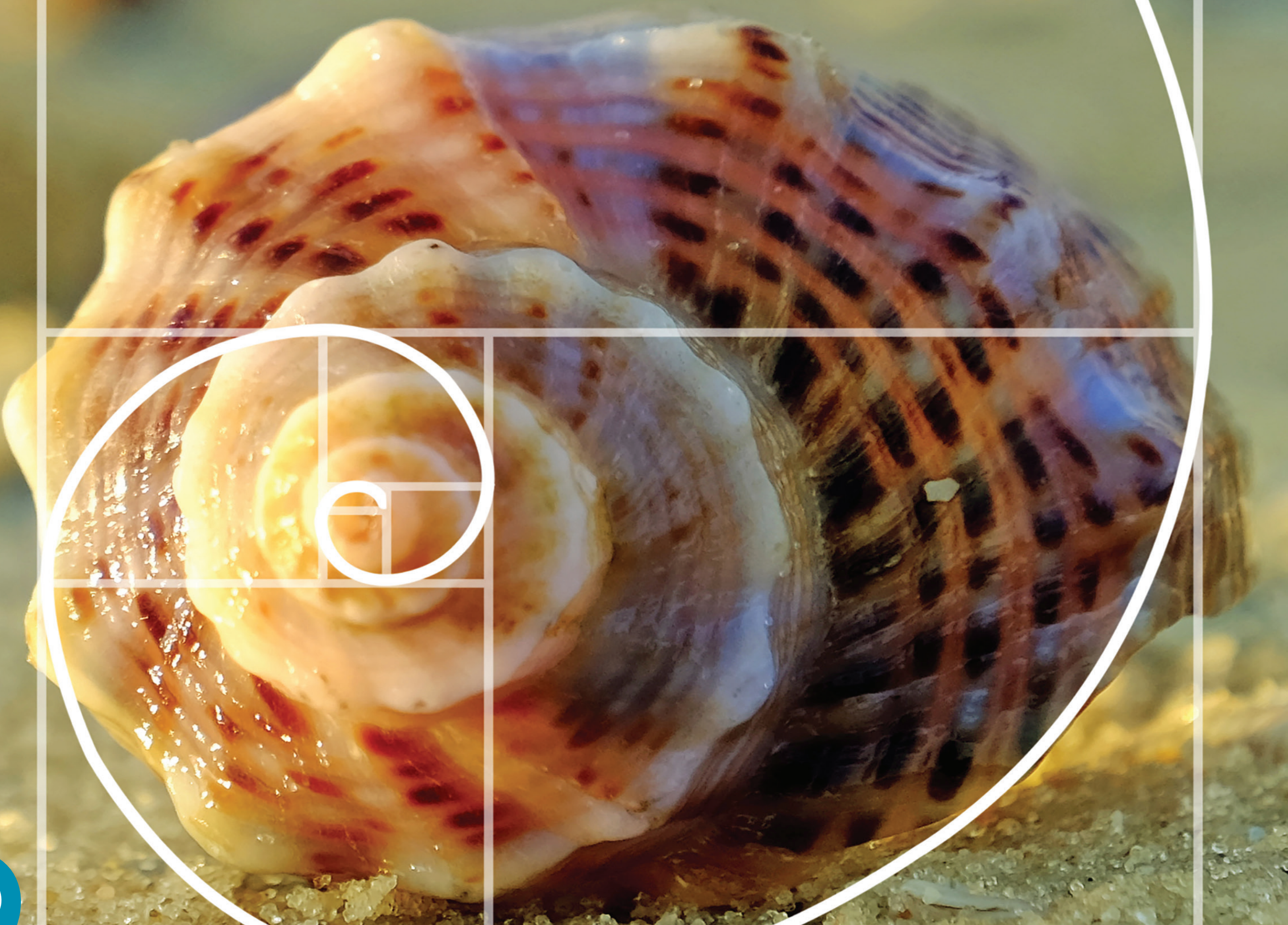
INTRODUCTORY MATHEMATICAL ANALYSIS

FOR BUSINESS, ECONOMICS, AND THE LIFE AND SOCIAL SCIENCES



INTRODUCTORY MATHEMATICAL ANALYSIS

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FOR BUSINESS, ECONOMICS, AND THE LIFE AND SOCIAL SCIENCES

FOURTEENTH EDITION
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Preface

The fourteenth edition of *Introductory Mathematical Analysis for Business, Economics, and the Life and Social Sciences (IMA)* continues to provide a mathematical foundation for students in a variety of fields and majors, as suggested by the title. As begun in the thirteenth edition, the book has three parts: College Algebra, Chapters 0–4; Finite Mathematics, Chapters 5–9; and Calculus, Chapters 10–17.

Schools that have two academic terms per year tend to give Business students a term devoted to Finite Mathematics and a term devoted to Calculus. For these schools we recommend Chapters 0 through 9 for the first course, starting wherever the preparation of the students allows, and Chapters 10 through 17 for the second, including as much as the students' background allows and their needs dictate.

For schools with three quarter or three semester courses per year there are a number of possible uses for this book. If their program allows three quarters of Mathematics, well-prepared Business students can start a first course on Finite Mathematics with Chapter 1 and proceed through topics of interest up to and including Chapter 9. In this scenario, a second course on Differential Calculus could start with Chapter 10 on Limits and Continuity, followed by the three "differentiation chapters", 11 through 13 inclusive. Here, Section 12.6 on Newton's Method can be omitted without loss of continuity, while some instructors may prefer to review Chapter 4 on Exponential and Logarithmic Functions prior to studying them as differentiable functions. Finally, a third course could comprise Chapters 14 through 17 on Integral Calculus with an introduction to Multivariable Calculus. Note that Chapter 16 is certainly not needed for Chapter 17 and Section 15.8 on Improper Integrals can be safely omitted if Chapter 16 is not covered.

Approach

Introductory Mathematical Analysis for Business, Economics, and the Life and Social Sciences (IMA) takes a unique approach to problem solving. As has been the case in earlier editions of this book, we establish an emphasis on algebraic calculations that sets this text apart from other introductory, applied mathematics books. The process of calculating with variables builds skill in mathematical modeling and paves the way for students to use calculus. The reader will not find a "definition-theorem-proof" treatment, but there is a sustained effort to impart a genuine mathematical treatment of applied problems. In particular, our guiding philosophy leads us to include informal proofs and general calculations that shed light on how the corresponding calculations are done in applied problems. Emphasis on developing algebraic skills is extended to the exercises, of which many, even those of the drill type, are given with general rather than numerical coefficients.

We have refined the organization of our book over many editions to present the content in very manageable portions for optimal teaching and learning. Inevitably, that process tends to put "weight" on a book, and the present edition makes a very concerted effort to pare the book back somewhat, both with respect to design features—making for a cleaner approach—and content—recognizing changing pedagogical needs.

Changes for the Fourteenth Edition

We continue to make the elementary notions in the early chapters pave the way for their use in more advanced topics. For example, while discussing factoring, a topic many students find somewhat arcane, we point out that the principle "$ab = 0$ implies $a = 0$ or $b = 0$", together with factoring, enables the splitting of some complicated equations into several simpler equations. We point out that percentages are just rescaled numbers via the "equation" $p\% = \frac{p}{100}$ so that, in calculus, "relative rate of change" and "percentage rate of change" are related by the "equation" $r = r \cdot 100\%$. We think that at this time, when negative interest rates are often discussed, even if seldom implemented, it is wise to be absolutely precise about simple notions that are often taken for granted. In fact, in the

Finance, Chapter 5, we explicitly discuss negative interest rates and ask, somewhat rhetorically, why banks do not use continuous compounding (given that for a long time now continuous compounding has been able to simplify calculations *in practice* as well as in theory).

Whenever possible, we have tried to incorporate the extra ideas that were in the "Explore and Extend" chapter-closers into the body of the text. For example, the functions tax rate $t(i)$ and tax paid $T(i)$ of income i, are seen for what they are: everyday examples of case-defined functions. We think that in the process of learning about polynomials it is helpful to include Horner's Method for their evaluation, since with even a simple calculator at hand this makes the calculation much faster. While doing linear programming, it sometimes helps to think of lines and planes, etcetera, in terms of intercepts alone, so we include an exercise to show that if a line has (nonzero) intercepts x_0 and y_0 then its equation is given by

$$\frac{x}{x_0} + \frac{y}{y_0} = 1$$

and, moreover, (for positive x_0 and y_0) we ask for a geometric interpretation of the equivalent equation $y_0x + x_0y = x_0y_0$.

But, turning to our "paring" of the previous *IMA*, let us begin with Linear Programming. This is surely one of the most important topics in the book for Business students. We now feel that, while students should know about the possibility of *Multiple Optimum Solutions* and *Degeneracy and Unbounded Solutions*, they do not have enough time to devote an entire, albeit short, section to each of these. The remaining sections of Chapter 7 are already demanding and we now content ourselves with providing simple alerts to these possibilities that are easily seen geometrically. (The deleted sections were always tagged as "omittable".)

We think further that, in Integral Calculus, it is far more important for Applied Mathematics students to be adept at using tables to evaluate integrals than to know about *Integration by Parts* and *Partial Fractions*. In fact, these topics, of endless joy to some as recreational problems, do not seem to fit well into the general scheme of serious problem solving. It is a fact of life that an elementary function (in the technical sense) can easily fail to have an elementary antiderivative, and it seems to us that *Parts* does not go far enough to rescue this difficulty to warrant the considerable time it takes to master the technique. Since *Partial Fractions* ultimately lead to elementary antiderivatives for all *rational* functions, they *are* part of serious problem solving and a better case can be made for their inclusion in an applied textbook. However, it is vainglorious to do so without the inverse tangent function at hand and, by longstanding tacit agreement, applied calculus books do not venture into trigonometry.

After deleting the sections mentioned above, we reorganized the remaining material of the "integration chapters", 14 and 15, to rebalance them. The first concludes with the Fundamental Theorem of Calculus while the second is more properly "applied". We think that the formerly daunting Chapter 17 has benefited from deletion of *Implicit Partial Differentiation*, the *Chain Rule* for partial differentiation, and *Lines of Regression*. Since Multivariable Calculus is extremely important for Applied Mathematics, we hope that this more manageable chapter will encourage instructors to include it in their syllabi.

Examples and Exercises

Most instructors and students will agree that the key to an effective textbook is in the quality and quantity of the examples and exercise sets. To that end, more than 850 examples are worked out in detail. Some of these examples include a *strategy* box designed to guide students through the general steps of the solution before the specific solution is obtained. (See, for example, Section 14.3 Example 4.) In addition, an abundant number of diagrams (almost 500) and exercises (more than 5000) are included. Of the exercises, approximately 20 percent have been either updated or written completely anew. In each exercise set, grouped problems are usually given in increasing order of difficulty. In most exercise sets the problems progress from the basic mechanical drill-type to more interesting thought-provoking problems. The exercises labeled with a coloured exercise number correlate to a “Now Work Problem N” statement and example in the section.

Based on the feedback we have received from users of this text, the diversity of the applications provided in both the exercise sets and examples is truly an asset of this book. Many real applied problems with accurate data are included. Students do not need to look hard to see how the mathematics they are learning is applied to everyday or work-related situations. A great deal of effort has been put into producing a proper balance between drill-type exercises and problems requiring the integration and application of the concepts learned.

Pedagogy and Hallmark Features

- **Applications:** An abundance and variety of applications for the intended audience appear throughout the book so that students see frequently how the mathematics they are learning can be used. These applications cover such diverse areas as business, economics, biology, medicine, sociology, psychology, ecology, statistics, earth science, and archaeology. Many of these applications are drawn from literature and are documented by references, sometimes from the Web. In some, the background and context are given in order to stimulate interest. However, the text is self-contained, in the sense that it assumes no prior exposure to the concepts on which the applications are based. (See, for example, Chapter 15, Section 7, Example 2.)
- **Now Work Problem N:** Throughout the text we have retained the popular *Now Work Problem N* feature. The idea is that after a worked example, students are directed to an end-of-section problem (labeled with a colored exercise number) that reinforces the ideas of the worked example. This gives students an opportunity to practice what they have just learned. Because the majority of these keyed exercises are odd-numbered, students can immediately check their answer in the back of the book to assess their level of understanding.
- **Cautions:** Cautionary warnings are presented in very much the same way an instructor would warn students in class of commonly made errors. These appear in the margin, along with other explanatory notes and emphases.
- **Definitions, key concepts, and important rules and formulas:** These are clearly stated and displayed as a way to make the navigation of the book that much easier for the student. (See, for example, the Definition of Derivative in Section 11.1.)
- **Review material:** Each chapter has a review section that contains a list of important terms and symbols, a chapter summary, and numerous review problems. In addition, key examples are referenced along with each group of important terms and symbols.
- **Inequalities and slack variables:** In Section 1.2, when inequalities are introduced we point out that $a \leq b$ is equivalent to “there exists a non-negative number, s, such that $a + s = b$”. The idea is not deep but the pedagogical point is that *slack variables*, key to implementing the simplex algorithm in Chapter 7, should be familiar and not distract from the rather technical material in linear programming.
- **Absolute value:** It is common to note that $|a - b|$ provides the distance from a to b. In Example 4e of Section 1.4 we point out that “x is less than σ units from μ” translates as $|x - \mu| < \sigma$. In Section 1.4 this is but an exercise with the notation, as it should be, but the point here is that later (in Chapter 9) μ will be the mean and σ the standard deviation of a random variable. Again we have separated, in advance, a simple idea from a more advanced one. Of course, Problem 12 of Problems 1.4, which asks the student to set up $|f(x) - L| < \epsilon$, has a similar agenda to Chapter 10 on limits.
- **Early treatment of summation notation:** This topic is necessary for study of the definite integral in Chapter 14, but it is *useful* long before that. Since it is a notation that is new to most students at this level, but no more than a notation, we get it out of the way in Chapter 1. By using it when convenient, *before coverage of the definite integral*, it is not a distraction from that challenging concept.

- **Section 1.6 on sequences:** This section provides several pedagogical advantages. The very definition is stated in a fashion that paves the way for the more important and more basic definition of function in Chapter 2. In summing the terms of a sequence we are able to practice the use of summation notation introduced in the preceding section. The most obvious benefit though is that "sequences" allows us a better organization in the annuities section of Chapter 5. Both the present and the future values of an annuity are obtained by summing (finite) geometric sequences. Later in the text, sequences arise in the definition of the number e in Chapter 4, in Markov chains in Chapter 9, and in Newton's method in Chapter 12, so that a helpful unifying reference is obtained.
- **Sum of an infinite sequence:** In the course of summing the terms of a finite sequence, it is natural to raise the possibility of summing the terms of an infinite sequence. This is a nonthreatening environment in which to provide a first foray into the world of limits. We simply explain how certain infinite geometric sequences have well-defined sums and phrase the results in a way that creates a toehold for the introduction of limits in Chapter 10. These particular infinite sums enable us to introduce the idea of a perpetuity, first informally in the sequence section, and then again in more detail in a separate section in Chapter 5.
- **Section 2.8, Functions of Several Variables:** The introduction to functions of several variables appears in Chapter 2 because it is a topic that should appear long before Calculus. Once we have done some calculus there are particular ways to use calculus in the study of functions of several variables, but these aspects should not be confused with the basics that we use throughout the book. For example, "a-sub-n-angle-r" and "s-sub-n-angle-r" studied in the Mathematics of Finance, Chapter 5, are perfectly good functions of two variables, and Linear Programming seeks to optimize linear functions of several variables subject to linear constraints.
- **Leontief's input-output analysis in Section 6.7:** In this section we have separated various aspects of the total problem. We begin by describing what we call the Leontief matrix A as an encoding of the input and output relationships between sectors of an economy. Since this matrix can often be assumed to be constant for a substantial period of time, we begin by assuming that A is a given. The simpler problem is then to determine the production, X, which is required to meet an external demand, D, for an economy whose Leontief matrix is A. We provide a careful account of this as the solution of $(I-A)X = D$. Since A can be assumed to be fixed while various demands, D, are investigated, there is *some* justification to compute $(I-A)^{-1}$ so that we have $X = (I-A)^{-1}D$. However, use of a matrix inverse should not be considered an essential part of the solution. Finally, we explain how the Leontief matrix can be found from a table of data that might be available to a planner.
- **Birthday probability in Section 8.4:** This is a treatment of the classic problem of determining the probability that at least 2 of n people have their birthday on the same day. While this problem is given as an example in many texts, the recursive formula that we give for calculating the probability as a function of n is not a common feature. It is reasonable to include it in this book because recursively defined sequences appear explicitly in Section 1.6.
- **Markov Chains:** We noticed that considerable simplification of the problem of finding steady state vectors is obtained by writing state vectors as columns rather than rows. This does necessitate that a transition matrix $\mathbf{T} = [t_{ij}]$ have t_{ij} ="probability that next state is i given that current state is j" but avoids several artificial transpositions.
- **Sign Charts for a function in Chapter 10:** The sign charts that we introduced in the 12th edition now make their appearance in Chapter 10. Our point is that these charts can be made for any real-valued function of a real variable and their help in graphing a function begins prior to the introduction of derivatives. Of course we continue to exploit their use in Chapter 13 "Curve Sketching" where, for each function f, we advocate making a sign chart for each of f, f', and f'', interpreted for f itself. When this is possible, the graph of the function becomes almost self-evident. We freely acknowledge that this is a blackboard technique used by many instructors, but it appears too rarely in textbooks.

Supplements

- **MyLab Math Online Course (access code required)** Built around Pearson's best-selling content, MyLab™ Math, is an online homework, tutorial, and assessment program designed to work with this text to engage students and improve results. MyLab Math can be successfully implemented in any classroom environment—lab-based, hybrid, fully online, or traditional. By addressing instructor and student needs, MyLab Math improves student learning. Used by more than 37 million students worldwide, MyLab Math delivers consistent, measurable gains in student learning outcomes, retention and subsequent course success. Visit www.mymathlab.com/results to learn more.

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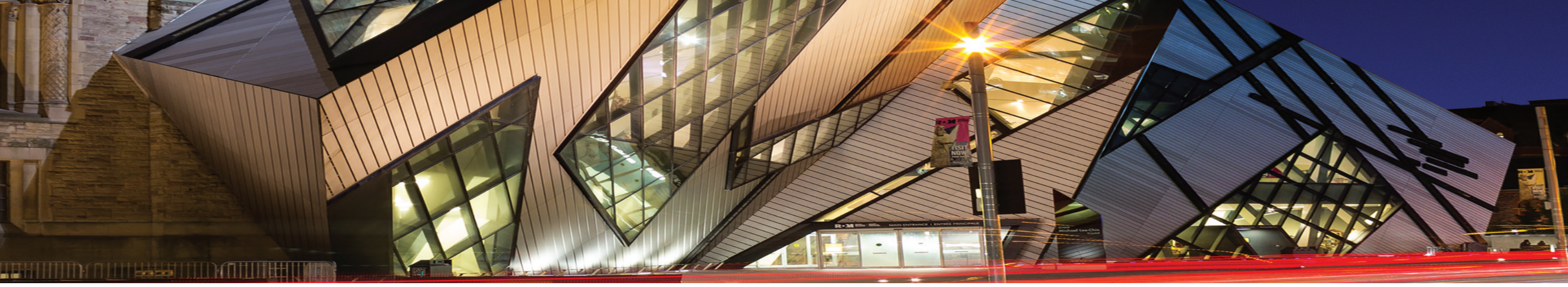

0 Review of Algebra

Lesley Griffith worked for a yacht supply company in Antibes, France. Often, she needed to examine receipts in which only the total paid was reported and then determine the amount of the total which was French "value-added tax". It is known as TVA for "Taxe à la Value Ajouté". The French TVA rate was 19.6% (but in January of 2014 it increased to 20%). A lot of Lesley's business came from Italian suppliers and purchasers, so she also had to deal with the similar problem of receipts containing Italian sales tax at 18% (now 22%).

A problem of this kind demands a formula, so that the user can just plug in a tax rate like 19.6% or 22% to suit a particular place and time, but many people are able to work through a particular case of the problem, using specified numbers, without knowing the formula. Thus, if Lesley had a 200-Euro French receipt, she might have reasoned as follows: If the item cost 100 Euros before tax, then the receipt total would be for 119.6 Euros with tax of 19.6, so *tax in a receipt total of 200 is to 200 as 19.6 is to 119.6*. Stated mathematically,

$$\frac{\text{tax in } 200}{200} = \frac{19.6}{119.6} \approx 0.164 = 16.4\%$$

If her reasoning is correct then the amount of TVA in a 200-Euro receipt is about 16.4% of 200 Euros, which is 32.8 Euros. In fact, many people will now guess that

$$\text{tax in } R = R\left(\frac{p}{100+p}\right)$$

gives the tax in a receipt R, when the tax rate is $p\%$. Thus, if Lesley felt confident about her deduction, she could have multiplied her Italian receipts by $\frac{18}{118}$ to determine the tax they contained.

Of course, most people do not remember formulas for very long and are uncomfortable basing a monetary calculation on an assumption such as the one we italicized above. There are lots of relationships that are more complicated than simple proportionality! The purpose of this chapter is to review the algebra necessary for you to construct your own formulas, *with confidence,* as needed. In particular, we will derive Lesley's formula from principles with which everybody is familiar. This usage of algebra will appear throughout the book, in the course of making *general calculations with variable quantities*.

In this chapter we will review real numbers and algebraic expressions and the basic operations on them. The chapter is designed to provide a brief review of some terms and methods of symbolic calculation. Probably, you have seen most of this material before. However, because these topics are important in handling the mathematics that comes later, an immediate second exposure to them may be beneficial. Devote whatever time is necessary to the sections in which you need review.

Objective

To become familiar with sets, in particular sets of real numbers, and the real-number line.

0.1 Sets of Real Numbers

A **set** is a collection of objects. For example, we can speak of the set of even numbers between 5 and 11, namely, 6, 8, and 10. An object in a set is called an **element** of that set. If this sounds a little circular, don't worry. The words *set* and *element* are like *line* and *point* in geometry. We cannot define them in more primitive terms. It is only with practice in using them that we come to understand their meaning. The situation is also rather like the way in which a child learns a first language. Without knowing *any* words, a child infers the meaning of a few very simple words by watching and listening to a parent and ultimately uses these very few words to build a working vocabulary. None of us needs to understand the mechanics of this process in order to learn how to speak. In the same way, it is possible to learn practical mathematics without becoming embroiled in the issue of undefined primitive terms.

One way to specify a set is by listing its elements, in any order, inside braces. For example, the previous set is $\{6, 8, 10\}$, which we could denote by a letter such as A, allowing us to write $A = \{6, 8, 10\}$. Note that $\{8, 10, 6\}$ also denotes the same set, as does $\{6, 8, 10, 10\}$. *A set is determined by its elements*, and neither rearrangements nor repetitions in a listing affect the set. A set A is said to be a subset of a set B if and only if every element of A is also an element of B. For example, if $A = \{6, 8, 10\}$ and $B = \{6, 8, 10, 12\}$, then A is a subset of B but B is not a subset of A. There is exactly one set which contains *no* elements. It is called *the empty set* and is denoted by $\emptyset$.

Certain sets of numbers have special names. The numbers 1, 2, 3, and so on form the set of **positive integers**:

$$\textit{set of positive integers} = \{1, 2, 3, \ldots\}$$

The three dots are an informal way of saying that the listing of elements is unending and the reader is expected to generate as many elements as needed from the pattern.

The positive integers together with 0 and the **negative integers** $-1, -2, -3, \ldots,$ form the set of **integers**:

$$\textit{set of integers} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

The reason for $q \neq 0$ is that we cannot divide by zero.

Every integer is a rational number.

Every rational number is a real number.

The set of real numbers consists of all decimal numbers.

The set of **rational numbers** consists of numbers, such as $\frac{1}{2}$ and $\frac{5}{3}$, that can be written as a quotient of two integers. That is, a rational number is a number that can be written as $\frac{p}{q}$, where p and q are integers and $q \neq 0$. (The symbol "$\neq$" is read "is not equal to.") For example, the numbers $\frac{19}{20}$, $\frac{-2}{7}$, and $\frac{-6}{-2}$ are rational. We remark that $\frac{2}{4}$, $\frac{1}{2}$, $\frac{3}{6}$, $\frac{-4}{-8}$, 0.5, and 50% all represent the same rational number. The integer 2 is rational, since $2 = \frac{2}{1}$. In fact, every integer is rational.

All rational numbers can be represented by decimal numbers that *terminate,* such as $\frac{3}{4} = 0.75$ and $\frac{3}{2} = 1.5$, or by *nonterminating, repeating decimal numbers* (composed of a group of digits that repeats without end), such as $\frac{2}{3} = 0.666\ldots$, $\frac{-4}{11} = -0.3636\ldots$, and $\frac{2}{15} = 0.1333\ldots$. Numbers represented by *nonterminating, nonrepeating* decimals are called **irrational numbers**. An irrational number cannot be written as an integer divided by an integer. The numbers π (pi) and $\sqrt{2}$ are examples of irrational numbers. Together, the rational numbers and the irrational numbers form the set of **real numbers**.

Real numbers can be represented by points on a line. First we choose a point on the line to represent zero. This point is called the *origin*. (See Figure 0.1.) Then a standard measure of distance, called a *unit distance,* is chosen and is successively marked off both to the right and to the left of the origin. With each point on the line we associate a directed distance, which depends on the position of the point with respect to the origin.

Some Points and Their Coordinates

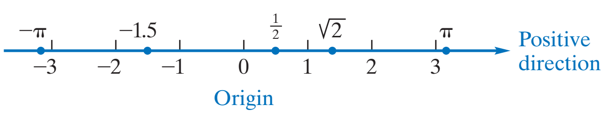

FIGURE 0.1 The real-number line.

Positions to the right of the origin are considered positive (+) and positions to the left are negative (−). For example, with the point $\frac{1}{2}$ unit to the right of the origin there corresponds the number $\frac{1}{2}$, which is called the **coordinate** of that point. Similarly, the coordinate of the point 1.5 units to the left of the origin is −1.5. In Figure 0.1, the coordinates of some points are marked. The arrowhead indicates that the direction to the right along the line is considered the positive direction.

To each point on the line there corresponds a unique real number, and to each real number there corresponds a unique point on the line. There is a *one-to-one correspondence* between points on the line and real numbers. We call such a line, with coordinates marked, a **real-number line**. We feel free to treat real numbers as points on a real-number line and vice versa.

EXAMPLE 1 Identifying Kinds of Real Numbers

Is it true that 0.151515 . . . is an irrational number?

Solution: The dots in 0.151515 . . . are understood to convey repetition of the digit string “15”. Irrational numbers were defined to be real numbers that are represented by a *nonterminating, nonrepeating* decimal, so 0.151515 . . . is not irrational. It is therefore a rational number. It is not immediately clear how to represent 0.151515 . . . as a quotient of integers. In Chapter 1 we will learn how to show that $0.151515\ldots = \frac{5}{33}$. You can check that this is *plausible* by entering $5 \div 33$ on a calculator, but you should also think about why the calculator exercise does not *prove* that $0.151515\ldots = \frac{5}{33}$.

Now Work Problem 7 ◁

PROBLEMS 0.1

In Problems 1–12, determine the truth of each statement. If the statement is false, give a reason why that is so.

1. $\sqrt{-13}$ is an integer.
2. $\frac{-2}{7}$ is rational.
3. −3 is a positive integer.
4. 0 is not rational.
5. $\sqrt{3}$ is rational.
6. $\frac{-1}{0}$ is a rational number.
7. $\sqrt{25}$ is not a positive integer.
8. $\sqrt{2}$ is a real number.
9. $\frac{0}{0}$ is rational.
10. π is a positive integer.
11. 0 is to the right of $-\sqrt{2}$ on the real-number line.
12. Every integer is positive or negative.
13. Every terminating decimal number can be regarded as a repeating decimal number.
14. $\sqrt{-1}$ is a real number.

Objective

To name, illustrate, and relate properties of the real numbers and their operations.

0.2 Some Properties of Real Numbers

We now state a few important properties of the real numbers. Let a, b, and c be real numbers.

1. The Transitive Property of Equality

If $a = b$ and $b = c$, then $a = c$.

Thus, two numbers that are both equal to a third number are equal to each other. For example, if $x = y$ and $y = 7$, then $x = 7$.

2. The Closure Properties of Addition and Multiplication

For all real numbers a and b, there are unique real numbers $a + b$ and ab.

This means that any two numbers can be added and multiplied, and the result in each case is a real number.

3. The Commutative Properties of Addition and Multiplication

$$a + b = b + a \quad \text{and} \quad ab = ba$$

This means that two numbers can be added or multiplied in any order. For example, $3 + 4 = 4 + 3$ and $(7)(-4) = (-4)(7)$.

4. The Associative Properties of Addition and Multiplication

$$a + (b + c) = (a + b) + c \quad \text{and} \quad a(bc) = (ab)c$$

This means that, for both addition and multiplication, numbers can be grouped in any order. For example, $2 + (3 + 4) = (2 + 3) + 4$; in both cases, the sum is 9. Similarly, $2x + (x + y) = (2x + x) + y$, and observe that the right side more obviously simplifies to $3x + y$ than does the left side. Also, $(6 \cdot \frac{1}{3}) \cdot 5 = 6(\frac{1}{3} \cdot 5)$, and here the left side obviously reduces to 10, so the right side does too.

5. The Identity Properties

There are unique real numbers denoted 0 and 1 such that, for each real number a,

$$0 + a = a \quad \text{and} \quad 1a = a$$

6. The Inverse Properties

For each real number a, there is a unique real number denoted $-a$ such that

$$a + (-a) = 0$$

The number $-a$ is called the **negative** of a.

For example, since $6 + (-6) = 0$, the negative of 6 is -6. The negative of a number is not necessarily a negative number. For example, the negative of -6 is 6, since $(-6) + (6) = 0$. That is, the negative of -6 is 6, so we can write $-(-6) = 6$.

For each real number a, *except* 0, there is a unique real number denoted a^{-1} such that

$$a \cdot a^{-1} = 1$$

The number a^{-1} is called the **reciprocal** of a.

Zero does not have a reciprocal because there is no number that when multiplied by 0 gives 1. This is a consequence of $0 \cdot a = 0$ in 7. The Distributive Properties.

Thus, all numbers *except* 0 have a reciprocal. Recall that a^{-1} can be written $\frac{1}{a}$. For example, the reciprocal of 3 is $\frac{1}{3}$, since $3(\frac{1}{3}) = 1$. Hence, $\frac{1}{3}$ is the reciprocal of 3. The reciprocal of $\frac{1}{3}$ is 3, since $(\frac{1}{3})(3) = 1$. *The reciprocal of 0 is not defined*.

7. The Distributive Properties

$$a(b + c) = ab + ac \quad \text{and} \quad (b + c)a = ba + ca$$

$$0 \cdot a = 0 = a \cdot 0$$

For example, although $2(3+4) = 2(7) = 14$, we can also write

$$2(3+4) = 2(3) + 2(4) = 6 + 8 = 14$$

Similarly,

$$(2+3)(4) = 2(4) + 3(4) = 8 + 12 = 20$$

and

$$x(z+4) = x(z) + x(4) = xz + 4x$$

The distributive property can be extended to the form

$$a(b+c+d) = ab + ac + ad$$

In fact, it can be extended to sums involving any number of terms.

Subtraction is defined in terms of addition:

$$a - b \quad \text{means} \quad a + (-b)$$

where $-b$ is the negative of b. Thus, $6 - 8$ means $6 + (-8)$.

In a similar way, we define **division** in terms of multiplication. If $b \neq 0$, then

$$a \div b \quad \text{means} \quad a(b^{-1})$$

Usually, we write either $\frac{a}{b}$ or a/b for $a \div b$. Since $b^{-1} = \frac{1}{b}$,

$$\frac{a}{b} = a(b^{-1}) = a\left(\frac{1}{b}\right)$$

$\frac{a}{b}$ means a times the reciprocal of b.

Thus, $\frac{3}{5}$ means 3 times $\frac{1}{5}$, where $\frac{1}{5}$ is the reciprocal of 5. Sometimes we refer to $\frac{a}{b}$ as the *ratio* of a to b. We remark that since 0 does not have a reciprocal, **division by 0 is not defined**.

The following examples show some manipulations involving the preceding properties.

EXAMPLE 1 Applying Properties of Real Numbers

a. $x(y - 3z + 2w) = (y - 3z + 2w)x$, by the commutative property of multiplication.

b. By the associative property of multiplication, $3(4 \cdot 5) = (3 \cdot 4)5$. Thus, the result of multiplying 3 by the product of 4 and 5 is the same as the result of multiplying the product of 3 and 4 by 5. In either case, the result is 60.

c. Show that $a(b \cdot c) \neq (ab) \cdot (ac)$

Solution: To show the negation of a general statement, it suffices to provide a *counterexample*. Here, taking $a = 2$ and $b = 1 = c$, we see that that $a(b \cdot c) = 2$ while $(ab) \cdot (ac) = 4$.

Now Work Problem 9 ◁

EXAMPLE 2 Applying Properties of Real Numbers

a. Show that $2 - \sqrt{2} = -\sqrt{2} + 2$.

Solution: By the definition of subtraction, $2 - \sqrt{2} = 2 + (-\sqrt{2})$. However, by the commutative property of addition, $2 + (-\sqrt{2}) = -\sqrt{2} + 2$. Hence, by the transitive property of equality, $2 - \sqrt{2} = -\sqrt{2} + 2$. Similarly, it is clear that, for any a and b, we have

$$a - b = -b + a$$

b. Show that $(8 + x) - y = 8 + (x - y)$.

Solution: Beginning with the left side, we have

$$\begin{aligned} (8+x)-y &= (8+x)+(-y) && \text{definition of subtraction} \\ &= 8+(x+(-y)) && \text{associative property} \\ &= 8+(x-y) && \text{definition of subtraction} \end{aligned}$$

Hence, by the transitive property of equality,

$$(8+x)-y = 8+(x-y)$$

Similarly, for all a, b, and c, we have

$$(a+b)-c = a+(b-c)$$

c. Show that $3(4x+2y+8) = 12x+6y+24$.

Solution: By the distributive property,

$$3(4x+2y+8) = 3(4x)+3(2y)+3(8)$$

But by the associative property of multiplication,

$$3(4x) = (3\cdot 4)x = 12x \quad \text{and similarly} \quad 3(2y) = 6y$$

Thus, $3(4x+2y+8) = 12x+6y+24$

Now Work Problem 25 ◁

EXAMPLE 3 Applying Properties of Real Numbers

a. Show that $\dfrac{ab}{c} = a\left(\dfrac{b}{c}\right)$, for $c \neq 0$.

Solution: The restriction is necessary. Neither side of the equation is defined if $c = 0$. By the definition of division,

$$\frac{ab}{c} = (ab)\cdot\frac{1}{c} \text{ for } c \neq 0$$

But by the associative property,

$$(ab)\cdot\frac{1}{c} = a\left(b\cdot\frac{1}{c}\right)$$

However, by the definition of division, $b\cdot\dfrac{1}{c} = \dfrac{b}{c}$. Thus,

$$\frac{ab}{c} = a\left(\frac{b}{c}\right)$$

We can also show that $\dfrac{ab}{c} = \left(\dfrac{a}{c}\right)b$.

b. Show that $\dfrac{a+b}{c} = \dfrac{a}{c}+\dfrac{b}{c}$ *for* $c \neq 0$.

Solution: (Again the restriction is necessary but we won't always bother to say so.) By the definition of division and the distributive property,

$$\frac{a+b}{c} = (a+b)\frac{1}{c} = a\cdot\frac{1}{c}+b\cdot\frac{1}{c}$$

However,

$$a\cdot\frac{1}{c}+b\cdot\frac{1}{c} = \frac{a}{c}+\frac{b}{c}$$

Hence,

$$\frac{a+b}{c} = \frac{a}{c}+\frac{b}{c}$$

Now Work Problem 27 ◁

Finding the product of several numbers can be done by considering products of numbers taken just two at a time. For example, to find the product of x, y, and z, we could first multiply x by y and then multiply that product by z; that is, we find $(xy)z$. Alternatively, we could multiply x by the product of y and z; that is, we find $x(yz)$. The associative property of multiplication guarantees that both results are identical, regardless of how the numbers are grouped. Thus, it is not ambiguous to write xyz. This concept can be extended to more than three numbers and applies equally well to addition.

Not only should you be able to manipulate real numbers, you should also be aware of, and familiar with, the terminology involved. It will help you read the book, follow your lectures, and — most importantly — allow you to frame your questions when you have difficulties.

The following list states important properties of real numbers that you should study thoroughly. Being able to manipulate real numbers is essential to your success in mathematics. A numerical example follows each property. *All denominators are assumed to be different from zero* (but for emphasis we have been explicit about these restrictions).

Property	***Example(s)***
1. $a - b = a + (-b)$	$2 - 7 = 2 + (-7) = -5$
2. $a - (-b) = a + b$	$2 - (-7) = 2 + 7 = 9$
3. $-a = (-1)(a)$	$-7 = (-1)(7)$
4. $a(b + c) = ab + ac$	$6(7 + 2) = 6 \cdot 7 + 6 \cdot 2 = 54$
5. $a(b - c) = ab - ac$	$6(7 - 2) = 6 \cdot 7 - 6 \cdot 2 = 30$
6. $-(a + b) = -a - b$	$-(7 + 2) = -7 - 2 = -9$
7. $-(a - b) = -a + b$	$-(2 - 7) = -2 + 7 = 5$
8. $-(-a) = a$	$-(-2) = 2$
9. $a(0) = 0$	$2(0) = 0$
10. $(-a)(b) = -(ab) = a(-b)$	$(-2)(7) = -(2 \cdot 7) = 2(-7) = -14$
11. $(-a)(-b) = ab$	$(-2)(-7) = 2 \cdot 7 = 14$
12. $\frac{a}{1} = a$	$\frac{7}{1} = 7, \frac{-2}{1} = -2$
13. $\frac{a}{b} = a\left(\frac{1}{b}\right)$ for $b \neq 0$	$\frac{2}{7} = 2\left(\frac{1}{7}\right)$
14. $\frac{a}{-b} = -\frac{a}{b} = \frac{-a}{b}$ for $b \neq 0$	$\frac{2}{-7} = -\frac{2}{7} = \frac{-2}{7}$
15. $\frac{-a}{-b} = \frac{a}{b}$ for $b \neq 0$	$\frac{-2}{-7} = \frac{2}{7}$
16. $\frac{0}{a} = 0$ for $a \neq 0$	$\frac{0}{7} = 0$
17. $\frac{a}{a} = 1$ for $a \neq 0$	$\frac{2}{2} = 1, \frac{-5}{-5} = 1$
18. $a\left(\frac{b}{a}\right) = b$ for $a \neq 0$	$2\left(\frac{7}{2}\right) = 7$
19. $a \cdot \frac{1}{a} = 1$ for $a \neq 0$	$2 \cdot \frac{1}{2} = 1$
20. $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ for $b, d \neq 0$	$\frac{2}{3} \cdot \frac{4}{5} = \frac{2 \cdot 4}{3 \cdot 5} = \frac{8}{15}$
21. $\frac{ab}{c} = \left(\frac{a}{c}\right)b = a\left(\frac{b}{c}\right)$ for $c \neq 0$	$\frac{2 \cdot 7}{3} = \frac{2}{3} \cdot 7 = 2 \cdot \frac{7}{3}$

Property	***Example(s)***
22. $\dfrac{a}{bc} = \dfrac{a}{b} \cdot \dfrac{1}{c} = \dfrac{1}{b} \cdot \dfrac{a}{c}$ for $b, c \neq 0$	$\dfrac{2}{3 \cdot 7} = \dfrac{2}{3} \cdot \dfrac{1}{7} = \dfrac{1}{3} \cdot \dfrac{2}{7}$
23. $\dfrac{a}{b} = \dfrac{a}{b} \cdot \dfrac{c}{c} = \dfrac{ac}{bc}$ for $b, c \neq 0$	$\dfrac{2}{7} = \left(\dfrac{2}{7}\right)\left(\dfrac{5}{5}\right) = \dfrac{2 \cdot 5}{7 \cdot 5}$
24. $\dfrac{a}{b(-c)} = \dfrac{a}{(-b)c} = \dfrac{-a}{bc} = \dfrac{-a}{(-b)(-c)} = -\dfrac{a}{bc}$ for $b, c \neq 0$	$\dfrac{2}{3(-5)} = \dfrac{2}{(-3)(5)} = \dfrac{-2}{3(5)} = \dfrac{-2}{(-3)(-5)} = -\dfrac{2}{3(5)} = -\dfrac{2}{15}$
25. $\dfrac{a(-b)}{c} = \dfrac{(-a)b}{c} = \dfrac{ab}{-c} = \dfrac{(-a)(-b)}{-c} = -\dfrac{ab}{c}$ for $c \neq 0$	$\dfrac{2(-3)}{5} = \dfrac{(-2)(3)}{5} = \dfrac{2(3)}{-5} = \dfrac{(-2)(-3)}{-5} = -\dfrac{2(3)}{5} = -\dfrac{6}{5}$
26. $\dfrac{a}{c} + \dfrac{b}{c} = \dfrac{a+b}{c}$ for $c \neq 0$	$\dfrac{2}{9} + \dfrac{3}{9} = \dfrac{2+3}{9} = \dfrac{5}{9}$
27. $\dfrac{a}{c} - \dfrac{b}{c} = \dfrac{a-b}{c}$ for $c \neq 0$	$\dfrac{2}{9} - \dfrac{3}{9} = \dfrac{2-3}{9} = \dfrac{-1}{9}$
28. $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad+bc}{bd}$ for $b, d \neq 0$	$\dfrac{4}{5} + \dfrac{2}{3} = \dfrac{4 \cdot 3 + 5 \cdot 2}{5 \cdot 3} = \dfrac{22}{15}$
29. $\dfrac{a}{b} - \dfrac{c}{d} = \dfrac{ad-bc}{bd}$ for $b, d \neq 0$	$\dfrac{4}{5} - \dfrac{2}{3} = \dfrac{4 \cdot 3 - 5 \cdot 2}{5 \cdot 3} = \dfrac{2}{15}$
30. $\dfrac{\frac{a}{b}}{\frac{c}{d}} = \dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{a}{b} \cdot \dfrac{d}{c} = \dfrac{ad}{bc}$ for $b, c, d \neq 0$	$\dfrac{\frac{2}{3}}{\frac{7}{5}} = \dfrac{2}{3} \div \dfrac{7}{5} = \dfrac{2}{3} \cdot \dfrac{5}{7} = \dfrac{2 \cdot 5}{3 \cdot 7} = \dfrac{10}{21}$
31. $\dfrac{a}{\frac{b}{c}} = a \div \dfrac{b}{c} = a \cdot \dfrac{c}{b} = \dfrac{ac}{b}$ for $b, c \neq 0$	$\dfrac{2}{\frac{3}{5}} = 2 \div \dfrac{3}{5} = 2 \cdot \dfrac{5}{3} = \dfrac{2 \cdot 5}{3} = \dfrac{10}{3}$
32. $\dfrac{\frac{a}{b}}{c} = \dfrac{a}{b} \div c = \dfrac{a}{b} \cdot \dfrac{1}{c} = \dfrac{a}{bc}$ for $b, c = 0$	$\dfrac{\frac{2}{3}}{5} = \dfrac{2}{3} \div 5 = \dfrac{2}{3} \cdot \dfrac{1}{5} = \dfrac{2}{3 \cdot 5} = \dfrac{2}{15}$

Property 23 is particularly important and could be called the **fundamental principle of fractions**. It states that *multiplying or dividing both the numerator and denominator of a fraction by the same nonzero number results in a fraction that is equal to the original fraction*. Thus,

$$\frac{7}{\frac{1}{8}} = \frac{7 \cdot 8}{\frac{1}{8} \cdot 8} = \frac{56}{1} = 56$$

By Properties 28 and 23, we have

$$\frac{2}{5} + \frac{4}{15} = \frac{2 \cdot 15 + 5 \cdot 4}{5 \cdot 15} = \frac{50}{75} = \frac{2 \cdot 25}{3 \cdot 25} = \frac{2}{3}$$

We can also do this problem by converting $\frac{2}{5}$ and $\frac{4}{15}$ into fractions that have the same denominators and then using Property 26. The fractions $\frac{2}{5}$ and $\frac{4}{15}$ can be written with a common denominator of $5 \cdot 15$:

$$\frac{2}{5} = \frac{2 \cdot 15}{5 \cdot 15} \quad \text{and} \quad \frac{4}{15} = \frac{4 \cdot 5}{15 \cdot 5}$$

However, 15 is the *least* such common denominator and is called the *least common denominator* (LCD) of $\frac{2}{5}$ and $\frac{4}{15}$. Thus,

$$\frac{2}{5} + \frac{4}{15} = \frac{2 \cdot 3}{5 \cdot 3} + \frac{4}{15} = \frac{6}{15} + \frac{4}{15} = \frac{6+4}{15} = \frac{10}{15} = \frac{2}{3}$$

Similarly,

$$\begin{aligned} \frac{3}{8} - \frac{5}{12} &= \frac{3 \cdot 3}{8 \cdot 3} - \frac{5 \cdot 2}{12 \cdot 2} \qquad \text{LCD} = 24 \\ &= \frac{9}{24} - \frac{10}{24} = \frac{9-10}{24} \\ &= -\frac{1}{24} \end{aligned}$$

PROBLEMS 0.2

In Problems 1–10, determine the truth of each statement.

1. Every real number has a reciprocal.

2. The reciprocal of 6.6 is 0.151515 . . . .

3. The negative of 7 is $\frac{-1}{7}$.

4. $1(x \cdot y) = (1 \cdot x)(1 \cdot y)$

5. $-x + y = -y + x$

6. $(x+2)(4) = 4x + 8$

7. $\frac{x+3}{5} = \frac{x}{5} + 3$

8. $3\left(\frac{x}{4}\right) = \frac{3x}{4}$

9. $2(x \cdot y) = (2x) \cdot (2y)$

10. $x(4y) = 4xy$

In Problems 11–20, state which properties of the real numbers are being used.

11. $2(x+y) = 2x + 2y$

12. $(x + 5.2) + 0.7y = x + (5.2 + 0.7y)$

13. $2(3y) = (2 \cdot 3)y$

14. $\frac{a}{b} = \frac{1}{b} \cdot a$

15. $5(b-a) = (a-b)(-5)$

16. $y + (x+y) = (y+x) + y$

17. $\frac{5x-y}{7} = 1/7(5x-y)$

18. $5(4+7) = 5(7+4)$

19. $(2+a)b = 2b + ba$

20. $(-1)(-3+4) = (-1)(-3) + (-1)(4)$

In Problems 21–27, show that the statements are true by using properties of the real numbers.

21. $2x(y-7) = 2xy - 14x$

22. $\frac{x}{y}z = x\frac{z}{y}$

23. $(x+y)(2) = 2x + 2y$

24. $a(b + (c+d)) = a((d+b)+c)$

25. $x((2y+1)+3) = 2xy + 4x$

26. $(1+a)(b+c) = b + c + ab + ac$

27. Show that $(x - y + z)w = xw - yw + zw$.
[*Hint*: $b + c + d = (b+c) + d$.]

Simplify each of the following, if possible.

28. $-2 + (-4)$ **29.** $-a + b$ **30.** $6 + (-4)$

31. $7 - 2$ **32.** $\frac{3}{2^{-1}}$ **33.** $-5 - (-13)$

34. $-(-a) + (-b)$ **35.** $(-2)(9)$ **36.** $7(-9)$

37. $(-1.6)(-0.5)$ **38.** $19(-1)$ **39.** $\frac{-1}{\frac{-1}{a}}$

40. $-(-6+x)$ **41.** $-7(x)$ **42.** $-3(a-b)$

43. $-(-6+(-y))$ **44.** $-3 \div 3a$ **45.** $-9 \div (-27)$

46. $(-a) \div (-b)$ **47.** $3 + (3^{-1}9)$ **48.** $3(-2(3) + 6(2))$

49. $(-a)(-b)(-1)$ **50.** $(-12)(-12)$ **51.** $X(1)$

52. $-71(x-2)$ **53.** $4(5+x)$ **54.** $-(x-y)$

55. $0(-x)$ **56.** $8\left(\frac{1}{11}\right)$ **57.** $\frac{X}{1}$

58. $\dfrac{14x}{21y}$ **59.** $\dfrac{2x}{-2}$ **60.** $\dfrac{2}{3} \cdot \dfrac{1}{x}$

61. $\dfrac{a}{c}(3b)$ **62.** $5a + (7 - 5a)$ **63.** $\dfrac{-aby}{-ax}$

64. $\dfrac{a}{b} \cdot \dfrac{1}{c}$ **65.** $\dfrac{2}{x} \cdot \dfrac{5}{y}$ **66.** $\dfrac{1}{2} + \dfrac{1}{3}$

67. $\dfrac{x}{3a} + \dfrac{y}{a}$ **68.** $\dfrac{3}{10} - \dfrac{7}{15}$ **69.** $\dfrac{a}{b} + \dfrac{c}{b}$

70. $\dfrac{X}{\sqrt{5}} - \dfrac{Y}{\sqrt{5}}$ **71.** $\dfrac{3}{2} - \dfrac{1}{4} + \dfrac{1}{6}$ **72.** $\dfrac{3}{7} - \dfrac{5}{9}$

73. $\dfrac{6}{\frac{x}{y}}$ **74.** $\dfrac{\frac{l}{w}}{m}$ **75.** $\dfrac{\frac{-x}{y^2}}{\frac{z}{xy}}$

76. $\dfrac{7}{0}$ **77.** $\dfrac{0}{X}$, for $X \neq 0$ **78.** $\dfrac{0}{0}$

Objective

To review positive integral exponents, the zero exponent, negative integral exponents, rational exponents, principal roots, radicals, and the procedure of rationalizing the denominator.

Some authors say that 0^0 is not defined. However, $0^0 = 1$ is a consistent and often useful definition.

0.3 Exponents and Radicals

The product $x \cdot x \cdot x$ of 3 x's is abbreviated x^3. In general, for n a positive integer, x^n is the abbreviation for the product of n x's. The letter n in x^n is called the **exponent**, and x is called the **base**. More specifically, if n is a positive integer, we have

1. $x^n = \underbrace{x \cdot x \cdot x \cdot \ldots \cdot x}_{n \text{ factors}}$

2. $x^{-n} = \dfrac{1}{x^n} = \dfrac{1}{\underbrace{x \cdot x \cdot x \cdot \ldots \cdot x}_{n \text{ factors}}}$ for $x \neq 0$

3. $\dfrac{1}{x^{-n}} = x^n$ for $x \neq 0$

4. $x^0 = 1$

EXAMPLE 1 Exponents

a. $\left(\dfrac{1}{2}\right)^4 = \left(\dfrac{1}{2}\right)\left(\dfrac{1}{2}\right)\left(\dfrac{1}{2}\right)\left(\dfrac{1}{2}\right) = \dfrac{1}{16}$

b. $3^{-5} = \dfrac{1}{3^5} = \dfrac{1}{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3} = \dfrac{1}{243}$

c. $\dfrac{1}{3^{-5}} = 3^5 = 243$

d. $2^0 = 1, \pi^0 = 1, (-5)^0 = 1$

e. $x^1 = x$

Now Work Problem 5 ◁

If $r^n = x$, where n is a positive integer, then r is an nth **root** of x. Second roots, the case $n = 2$, are called square roots; and third roots, the case $n = 3$, are called cube roots. For example, $3^2 = 9$, so 3 is a square root of 9. Since $(-3)^2 = 9$, -3 is also a square root of 9. Similarly, -2 is a cube root of -8, since $(-2)^3 = -8$, while 5 is a fourth root of 625 since $5^4 = 625$.

Some numbers do not have an nth root that is a real number. For example, since the square of any real number is nonnegative: there is no real number that is a square root of -4.

The **principal *n*th root** of x is the nth root of x that is positive if x is positive and is negative if x is negative and n is odd. We denote the principal nth root of x by $\sqrt[n]{x}$. Thus,

$$\sqrt[n]{x} \text{ is } \begin{cases} \text{positive if } x \text{ is positive} \\ \text{negative if } x \text{ is negative and } n \text{ is odd} \end{cases}$$

For example, $\sqrt[2]{9} = 3$, $\sqrt[3]{-8} = -2$, and $\sqrt[3]{\dfrac{1}{27}} = \frac{1}{3}$. We define $\sqrt[n]{0} = 0$.

Although both 2 and -2 are square roots of 4, the principal square root of 4 is 2, not -2. Hence, $\sqrt{4} = 2$. For positive x, we often write $\pm\sqrt{x}$ to indicate both square roots of x, and "$\pm\sqrt{4} = \pm 2$" is a convenient short way of writing "$\sqrt{4} = 2$ and $-\sqrt{4} = -2$", but the only value of $\sqrt{4}$ is 2.

The symbol $\sqrt[n]{x}$ is called a **radical**. With principal square roots we usually write $\sqrt{x}$ instead of $\sqrt[2]{x}$. Thus, $\sqrt{9} = 3$.

If x is positive, the expression $x^{p/q}$, where p and q are integers with no common factors and q is positive, is defined to be $\sqrt[q]{x^p}$. Hence,

$$x^{3/4} = \sqrt[4]{x^3}; \; 8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$$

$$4^{-1/2} = \sqrt[2]{4^{-1}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

Here are the basic laws of exponents and radicals:

Law	***Example(s)***
1. $x^m \cdot x^n = x^{m+n}$	$2^3 \cdot 2^5 = 2^8 = 256; \; x^2 \cdot x^3 = x^5$
2. $x^0 = 1$	$2^0 = 1$
3. $x^{-n} = \dfrac{1}{x^n}$	$2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$
4. $\dfrac{1}{x^{-n}} = x^n$	$\dfrac{1}{2^{-3}} = 2^3 = 8; \; \dfrac{1}{x^{-5}} = x^5$
5. $\dfrac{x^m}{x^n} = x^{m-n} = \dfrac{1}{x^{n-m}}$	$\dfrac{2^{12}}{2^8} = 2^4 = 16; \; \dfrac{x^8}{x^{12}} = \dfrac{1}{x^4}$
6. $\dfrac{x^m}{x^m} = 1$	$\dfrac{2^4}{2^4} = 1$
7. $(x^m)^n = x^{mn}$	$(2^3)^5 = 2^{15}; \; (x^2)^3 = x^6$
8. $(xy)^n = x^n y^n$	$(2 \cdot 4)^3 = 2^3 \cdot 4^3 = 8 \cdot 64 = 512$
9. $\left(\dfrac{x}{y}\right)^n = \dfrac{x^n}{y^n}$	$\left(\dfrac{2}{3}\right)^3 = \dfrac{2^3}{3^3} = \dfrac{8}{27}$
10. $\left(\dfrac{x}{y}\right)^{-n} = \left(\dfrac{y}{x}\right)^n$	$\left(\dfrac{3}{4}\right)^{-2} = \left(\dfrac{4}{3}\right)^2 = \dfrac{16}{9}$
11. $x^{1/n} = \sqrt[n]{x}$	$3^{1/5} = \sqrt[5]{3}$
12. $x^{-1/n} = \dfrac{1}{x^{1/n}} = \dfrac{1}{\sqrt[n]{x}}$	$4^{-1/2} = \dfrac{1}{4^{1/2}} = \dfrac{1}{\sqrt{4}} = \dfrac{1}{2}$
13. $\sqrt[n]{x}\sqrt[n]{y} = \sqrt[n]{xy}$	$\sqrt[3]{9}\sqrt[3]{2} = \sqrt[3]{18}$
14. $\dfrac{\sqrt[n]{x}}{\sqrt[n]{y}} = \sqrt[n]{\dfrac{x}{y}}$	$\dfrac{\sqrt[3]{90}}{\sqrt[3]{10}} = \sqrt[3]{\dfrac{90}{10}} = \sqrt[3]{9}$
15. $\sqrt[m]{\sqrt[n]{x}} = \sqrt[mn]{x}$	$\sqrt[3]{\sqrt[4]{2}} = \sqrt[12]{2}$
16. $x^{m/n} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$	$8^{2/3} = \sqrt[3]{8^2} = (\sqrt[3]{8})^2 = 2^2 = 4$
17. $(\sqrt[m]{x})^m = x$	$(\sqrt[8]{7})^8 = 7$

EXAMPLE 2 Exponents and Radicals

When computing $x^{m/n}$, it is often easier to first find $\sqrt[n]{x}$ and then raise the result to the mth power. Thus, $(-27)^{4/3} = (\sqrt[3]{-27})^4 = (-3)^4 = 81$.

a. By Law 1,

$$x^6x^8 = x^{6+8} = x^{14}$$

$$a^3b^2a^5b = a^3a^5b^2b^1 = a^8b^3$$

$$x^{11}x^{-5} = x^{11-5} = x^6$$

$$z^{2/5}z^{3/5} = z^1 = z$$

$$xx^{1/2} = x^1x^{1/2} = x^{3/2}$$

b. By Law 16,

$$\left(\frac{1}{4}\right)^{3/2} = \left(\sqrt{\frac{1}{4}}\right)^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

c. $$\left(-\frac{8}{27}\right)^{4/3} = \left(\sqrt[3]{\frac{-8}{27}}\right)^4 = \left(\frac{\sqrt[3]{-8}}{\sqrt[3]{27}}\right)^4 \quad \text{Laws 16 and 14}$$

$$= \left(\frac{-2}{3}\right)^4$$

$$= \frac{(-2)^4}{3^4} = \frac{16}{81} \quad \text{Law 9}$$

d. $$(64a^3)^{2/3} = 64^{2/3}(a^3)^{2/3} \quad \text{Law 8}$$

$$= (\sqrt[3]{64})^2 a^2 \quad \text{Laws 16 and 7}$$

$$= (4)^2 a^2 = 16a^2$$

Now Work Problem 39 ◁

Rationalizing the numerator is a similar procedure.

Rationalizing the denominator of a fraction is a procedure in which a fraction having a radical in its denominator is expressed as an equal fraction without a radical in its denominator. We use the fundamental principle of fractions, as Example 3 shows.

EXAMPLE 3 Rationalizing Denominators

a. $$\frac{2}{\sqrt{5}} = \frac{2}{5^{1/2}} = \frac{2 \cdot 5^{1/2}}{5^{1/2} \cdot 5^{1/2}} = \frac{2 \cdot 5^{1/2}}{5^1} = \frac{2\sqrt{5}}{5}$$

b. $$\frac{2}{\sqrt[6]{3x^5}} = \frac{2}{\sqrt[6]{3} \cdot \sqrt[6]{x^5}} = \frac{2}{3^{1/6}x^{5/6}} = \frac{2 \cdot 3^{5/6}x^{1/6}}{3^{1/6}x^{5/6} \cdot 3^{5/6}x^{1/6}} \quad \text{for } x \neq 0$$

$$= \frac{2(3^5x)^{1/6}}{3x} = \frac{2\sqrt[6]{3^5x}}{3x}$$

Now Work Problem 63 ◁

The following examples illustrate various applications of the laws of exponents and radicals. All denominators are understood to be nonzero.

EXAMPLE 4 Exponents

a. Eliminate negative exponents in $\frac{x^{-2}y^3}{z^{-2}}$ for $x \neq 0, z \neq 0$.

Solution: $$\frac{x^{-2}y^3}{z^{-2}} = x^{-2} \cdot y^3 \cdot \frac{1}{z^{-2}} = \frac{1}{x^2} \cdot y^3 \cdot z^2 = \frac{y^3z^2}{x^2}$$

By comparing our answer with the original expression, we conclude that we can bring a factor of the numerator down to the denominator, and vice versa, by changing the sign of the exponent.

b. Simplify $\frac{x^2y^7}{x^3y^5}$ for $x \neq 0, y \neq 0$.

Solution: $$\frac{x^2y^7}{x^3y^5} = \frac{y^{7-5}}{x^{3-2}} = \frac{y^2}{x}$$

c. Simplify $(x^5y^8)^5$.

Solution: $$(x^5y^8)^5 = (x^5)^5(y^8)^5 = x^{25}y^{40}$$

d. Simplify $(x^{5/9}y^{4/3})^{18}$.

Solution: $$(x^{5/9}y^{4/3})^{18} = (x^{5/9})^{18}(y^{4/3})^{18} = x^{10}y^{24}$$

e. Simplify $\left(\frac{x^{1/5}y^{6/5}}{z^{2/5}}\right)^5$ for $z \neq 0$.

Solution: $$\left(\frac{x^{1/5}y^{6/5}}{z^{2/5}}\right)^5 = \frac{(x^{1/5}y^{6/5})^5}{(z^{2/5})^5} = \frac{xy^6}{z^2}$$

f. Simplify $\frac{x^3}{y^2} \div \frac{x^6}{y^5}$ for $x \neq 0, y \neq 0$.

Solution: $$\frac{x^3}{y^2} \div \frac{x^6}{y^5} = \frac{x^3}{y^2} \cdot \frac{y^5}{x^6} = \frac{y^3}{x^3}$$

Now Work Problem 51 ◁

EXAMPLE 5 Exponents

a. For $x \neq 0$ and $y \neq 0$, eliminate negative exponents in $x^{-1} + y^{-1}$ and simplify.

Solution: $$x^{-1} + y^{-1} = \frac{1}{x} + \frac{1}{y} = \frac{y+x}{xy}$$

b. Simplify $x^{3/2} - x^{1/2}$ by using the distributive law.

Solution: $$x^{3/2} - x^{1/2} = x^{1/2}(x-1)$$

c. For $x \neq 0$, eliminate negative exponents in $7x^{-2} + (7x)^{-2}$.

Solution: $$7x^{-2} + (7x)^{-2} = \frac{7}{x^2} + \frac{1}{(7x)^2} = \frac{7}{x^2} + \frac{1}{49x^2} = \frac{344}{49x^2}$$

d. For $x \neq 0$ and $y \neq 0$, eliminate negative exponents in $(x^{-1} - y^{-1})^{-2}$.

Solution: $$(x^{-1} - y^{-1})^{-2} = \left(\frac{1}{x} - \frac{1}{y}\right)^{-2} = \left(\frac{y-x}{xy}\right)^{-2}$$
$$= \left(\frac{xy}{y-x}\right)^2 = \frac{x^2y^2}{(y-x)^2}$$

e. Apply the distributive law to $x^{2/5}(y^{1/2} + 2x^{6/5})$.

Solution: $$x^{2/5}(y^{1/2} + 2x^{6/5}) = x^{2/5}y^{1/2} + 2x^{8/5}$$

Now Work Problem 41 ◁

EXAMPLE 6 Radicals

a. Simplify $\sqrt[4]{48}$.

Solution: $$\sqrt[4]{48} = \sqrt[4]{16 \cdot 3} = \sqrt[4]{16}\,\sqrt[4]{3} = 2\sqrt[4]{3}$$

b. Rewrite $\sqrt{2+5x}$ without using a radical sign.

Solution: $$\sqrt{2+5x} = (2+5x)^{1/2}$$

c. Rationalize the denominator of $\frac{\sqrt[5]{2}}{\sqrt[3]{6}}$ and simplify.

Solution: $$\frac{\sqrt[5]{2}}{\sqrt[3]{6}} = \frac{2^{1/5} \cdot 6^{2/3}}{6^{1/3} \cdot 6^{2/3}} = \frac{2^{3/15}6^{10/15}}{6} = \frac{(2^3 6^{10})^{1/15}}{6} = \frac{\sqrt[15]{2^3 6^{10}}}{6}$$

d. Simplify $\dfrac{\sqrt{20}}{\sqrt{5}}$.

Solution: $$\frac{\sqrt{20}}{\sqrt{5}} = \sqrt{\frac{20}{5}} = \sqrt{4} = 2$$

Now Work Problem 71 ◁

EXAMPLE 7 **Radicals**

a. Simplify $\sqrt[3]{x^6y^4}$.

Solution: $$\sqrt[3]{x^6y^4} = \sqrt[3]{(x^2)^3y^3y} = \sqrt[3]{(x^2)^3} \cdot \sqrt[3]{y^3} \cdot \sqrt[3]{y}$$
$$= x^2y\sqrt[3]{y}$$

b. Simplify $\sqrt{\dfrac{2}{7}}$.

Solution: $$\sqrt{\frac{2}{7}} = \sqrt{\frac{2 \cdot 7}{7 \cdot 7}} = \sqrt{\frac{14}{7^2}} = \frac{\sqrt{14}}{\sqrt{7^2}} = \frac{\sqrt{14}}{7}$$

c. Simplify $\sqrt{250} - \sqrt{50} + 15\sqrt{2}$.

Solution: $$\sqrt{250} - \sqrt{50} + 15\sqrt{2} = \sqrt{25 \cdot 10} - \sqrt{25 \cdot 2} + 15\sqrt{2}$$
$$= 5\sqrt{10} - 5\sqrt{2} + 15\sqrt{2}$$
$$= 5\sqrt{10} + 10\sqrt{2}$$

d. If x is any real number, simplify $\sqrt{x^2}$.

Solution: $$\sqrt{x^2} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Thus, $\sqrt{2^2} = 2$ and $\sqrt{(-3)^2} = -(-3) = 3$.

Now Work Problem 75 ◁

PROBLEMS 0.3

In Problems 1–14, simplify and express all answers in terms of positive exponents.

1. $(2^3)(2^2)$ **2.** x^6x^9 **3.** $17^5 \cdot 17^2$

4. z^3zz^2 **5.** $\dfrac{x^3x^5}{y^9y^5}$ **6.** $(x^{12})^4$

7. $\dfrac{(a^3)^7}{(b^4)^5}$ **8.** $\left(\dfrac{13^{14}}{13}\right)^2$ **9.** $(2x^2y^3)^3$

10. $\left(\dfrac{w^2s^3}{y^2}\right)^2$ **11.** $\dfrac{x^9}{x^5}$ **12.** $\left(\dfrac{2a^4}{7b^5}\right)^6$

13. $\dfrac{(y^3)^4}{(y^2)^3y^2}$ **14.** $\dfrac{(x^2)^3(x^3)^2}{(x^3)^4}$

In Problems 15–28, evaluate the expressions.

15. $\sqrt{25}$ **16.** $\sqrt[4]{81}$ **17.** $\sqrt[7]{-128}$

18. $\sqrt[5]{0.00243}$ **19.** $\sqrt[4]{\dfrac{1}{16}}$ **20.** $\sqrt[3]{-\dfrac{8}{27}}$

21. $(49)^{1/2}$ **22.** $(64)^{1/3}$ **23.** $81^{3/4}$

24. $(9)^{-5/2}$ **25.** $(32)^{-2/5}$ **26.** $(0.09)^{-1/2}$

27. $\left(\dfrac{1}{32}\right)^{4/5}$ **28.** $\left(-\dfrac{243}{1024}\right)^{2/5}$

In Problems 29–40, simplify the expressions.

29. $\sqrt{50}$ **30.** $\sqrt[3]{54}$ **31.** $\sqrt[3]{2x^3}$

32. $\sqrt{4x}$ **33.** $\sqrt{49u^8}$ **34.** $\sqrt[4]{\dfrac{x}{16}}$

35. $2\sqrt{8} - 5\sqrt{27} + \sqrt[3]{128}$ **36.** $\sqrt{\dfrac{3}{13}}$

37. $(9z^4)^{1/2}$ **38.** $(729x^6)^{3/2}$

39. $\left(\dfrac{27t^3}{8}\right)^{2/3}$ **40.** $\left(\dfrac{256}{x^{12}}\right)^{-3/4}$

In Problems 41–52, write the expressions in terms of positive exponents only. Avoid all radicals in the final form. For example,

$$y^{-1}\sqrt{x} = \frac{x^{1/2}}{y}$$

41. $\dfrac{a^5b^{-3}}{c^2}$ **42.** $\sqrt[5]{x^2y^3z^{-10}}$ **43.** $3a^{-1}b^{-2}c^{-3}$